ELECTRIC FORCES ON A CIRCULAR CYLINDRICAL INCLUSION AT THE SURFACE SEPARATING TWO DIELECTRICS

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1. Introduction. Mechanical forces operate on the surface of dielectrics in an electric field. These forces are electrical and exist even if there are no free charges because of polarization phenomena. These forces are proportional to the gradient of the dielectric permeability and the square of the electric field strength. Therefore they are largest in regions where the dielectric permeability changes sharply and where strong electric fields arise. These conditions occur most often on surfaces of artificial materials which have fissures, voids, and various inhomogeneities.

Here we study the force distribution on a plane surface separating dielectrics with different properties, when there is an inclusion on the separating surface. In this manner we examine forces which cause polarization effects in a three-component, piecewise homogeneous medium. The cylindrical inclusion, which has a circular cross section, is parallel and tangent to the boundary surface. The external electric field is perpendicular to the axis of the cylinder.

This system is characteristic of several insulator designs and, moreover, serves as a theoretical model for describing electrophysical processes in several devices (separators, for example) in which an electric field acts on dispersed materials.

The presence of an inclusion at the boundary surface leads to a local inhomogeneity in the electric field, which in turn creates a nonuniform force distribution. Here the inclusion is subject to the action of an integral force which tends either to "clamp" the inclusion to the surface or else to "expel" it from the surface. The direction in which the integral force acts depends on the relationship among the dielectric permeabilities of all materials which form this system.

The electric field must be calculated before the forces are. In this case, the theory of functions of a complex variable can be used to solve the field problem. This method allows us to obtain an exact solution to the boundary problem and then to calculate the force distribution in the system for general relative dielectric material characteristics.

2. Boundary Problem. On the boundary surface between two different dielectric materials let there be an inclusion with a dielectric permeability different from that of the first two materials. It is assumed that the radius of the regions occupied by the first two materials greatly exceeds the radius of the cylinder. In this case the following calculational model can be used to study electric phenomena near the inclusion. The first two materials occupy half spaces which are separated by a plane; the cylindrical inclusion is tangent to this plane (Fig. 1a). Consequently, we are looking at a three-component, piecewise homogeneous material which occupies all space. This whole system is located in a homogeneous electric field which is perpendicular to the axis of the cylindrical inclusion. Under these conditions, the electric field is planeparallel in the system.

If there are no free charges in each of the three materials, the stationary electric field satisfies the equation

$$\operatorname{rot}\mathbf{E}=0,\ \operatorname{div}\mathbf{D}=0,\ \mathbf{D}=\varepsilon\mathbf{E}, \tag{2.1}$$

where E is the electric field strength vector; **D** is the dielectric field displacement vector; and ε is the dielectric permeability.

The two-dimensional Eqs. (2.1) allow us to introduce functions which are holomorphic in the plane of the variables (x, y):

$$E(z)=E_x-iE_y,\ D(z)=D_x-iD_y\quad (z=x+iy). \tag{2.2}$$

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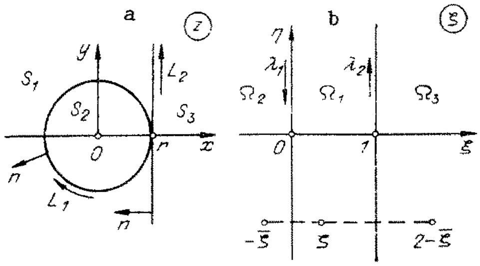

Fig. 1

The vectors **E** and **D** are related to the functions (2.2) by the complex conjugate operator, which we denote by a bar:

$$\mathbf{E} = \overline{E(z)}, \quad \mathbf{D} = \overline{D(z)}.$$

The dielectric ε_p is constant within each of the three regions S_p ($p = 1, 2, 3$) but differs from one region to another.

If there is purely electrical contact at the boundary of the different dielectrics, the normal component of the vector **D** and the tangential component of the vector **E** are continuous across the boundary surface. Then, from Eq. (2.1), we write these conditions in the form

$$\begin{gathered} \operatorname{Re}\{n(t)\varepsilon_1 E_1(t)\} = \operatorname{Re}\{n(t)\varepsilon_m E_m(t)\}, \\ \operatorname{Im}\{n(t)E_1(t)\} = \operatorname{Im}\{n(t)E_m(t)\}, \end{gathered} \quad t \in L_{m-1}, \quad m = 2, 3. \tag{2.3}$$

Here L_1 and L_2 are the directed boundary contours which separate the plane z into regions S_p (Fig. 1a), and $n(t)$ is the unit normal

$$n(t) = t/r, \ t \in L_1; \ n(t) = -1, \ t \in L_2, \tag{2.4}$$

where r is the radius of the circular inclusion.

The external electric field is specified by its value at a point an infinite distance away (as it is approached from region S_1):

$$E_1(\infty) = E_0 = E_{0x} - iE_{0y} = \text{const.} \tag{2.5}$$

If the infinitely removed point is approached from region S_3, then

$$E_3(\infty) = \frac{1}{2\varepsilon_3}\left[(\varepsilon_1 + \varepsilon_3)E_0 + (\varepsilon_1 - \varepsilon_3)\bar{E}_0\right]. \tag{2.6}$$

Equation (2.6) follows from Eqs. (2.3) and (2.4) for $m = 3$, with a consideration of the condition (2.5).

Thus, finding the function $E(z) = \{E_1(z), E_2(z), E_3(z)\}$ is reduced to the boundary problem [(2.3) and (2.4)] and one of two equivalent auxiliary conditions (2.5) or (2.6). Based on physical considerations, the function $E(z)$ at $z = r$ can be represented as an integrable singularity.

We use (2.4) to write the boundary conditions (2.3) in inverse form

$$\begin{aligned} &\varepsilon_1 E_1(t) + \varepsilon_1\left(\frac{r}{t}\right)^2 \overline{E_1(t)} = \varepsilon_2 E_2(t) + \varepsilon_2\left(\frac{r}{t}\right)^2 \overline{E_2(t)}, \\ &E_1(t) - \left(\frac{r}{t}\right)^2 \overline{E_1(t)} = E_2(t) - \left(\frac{r}{t}\right)^2 \overline{E_2(t)}, \end{aligned} \quad t \in L_1;$$
$$\begin{aligned} &\varepsilon_1 E_1(t) + \varepsilon_1 \overline{E_1(t)} = \varepsilon_3 E_3(t) + \varepsilon_3 \overline{E_3(t)}, \\ &E_1(t) - \overline{E_1(t)} = E_3(t) - \overline{E_3(t)}. \end{aligned} \quad t \in L_2. \tag{2.7}$$

Here it is assumed that $\bar{t} = r^2/t$ for $t \in L_1$ and thus that $\overline{n(t)} = r/t$.

One of the functions, for example $\overline{E_1(t)}$, can be excluded from each pair of Eqs. (2.7); then

$$2\varepsilon_1 E_1(t) = (\varepsilon_1 + \varepsilon_2)E_2(t) - (\varepsilon_1 - \varepsilon_2)\left(\frac{r}{t}\right)^2 \overline{E_2(t)}, \ t \in L_1; \tag{2.8}$$
$$2\varepsilon_1 E_1(t) = (\varepsilon_1 + \varepsilon_3)E_3(t) - (\varepsilon_1 - \varepsilon_3)\overline{E_3(t)}. \qquad t \in L_2.$$

In future calculations it is convenient to operate with relative dielectric permeabilities, which we define as follows:

$$\Delta_{1m} = \frac{\varepsilon_1 - \varepsilon_m}{\varepsilon_1 + \varepsilon_m}, \quad -1 < \Delta_{1m} < 1 \ (m = 2,\ 3). \tag{2.9}$$

By using Eqs. (2.9), the boundary relationships (2.8) can be written in an equivalent form

$$\begin{aligned} (1 + \Delta_{12}) E_1(t) &= E_2(t) - \Delta_{12}\left(\frac{r}{t}\right)^2 \overline{E_2(t)}, \quad t \in L_1; \\ (1 + \Delta_{13}) E_1(t) &= E_3(t) - \Delta_{13}\overline{E_3(t)}, \qquad t \in L_2, \end{aligned} \tag{2.10}$$

and the condition (2.5) or (2.6) at infinity takes the form

$$E_1(\infty) = E_0 \quad \text{or} \quad E_3(\infty) = \frac{E_0 + \Delta_{13}\bar{E}_0}{1 - \Delta_{13}}. \tag{2.11}$$

Thus, the calculation of the electric field in this inhomogeneous system is reduced to solving Eqs. (2.10) and (2.11), which are a particular case of the generalized Riemann boundary problem (now this problem is often called the R-linear conjugation problem).

3. Electric Field. The boundary problem formulated in the previous section has an exact solution, which can be obtained by the method of conformal mapping and the principle of analytic continuation. The solution to Eqs. (2.10) and (2.11) is discussed in the Appendix. The final results are presented below.

General Solution. The components of the electric field in the system of coordinates shown in Fig. 1a are defined as follows:

$$\begin{aligned} E_1(z) &= E_0 + r^2 \sum_{k=1}^{\infty} \left\{ \frac{(\Delta_{12}\Delta_{13})^k}{k^2} \left[E_0 \left(z - \frac{k+1}{k} r \right)^{-2} - \frac{\bar{E}_0}{\Delta_{13}} \left(z - \frac{k-1}{k} r \right)^{-2} \right] \right\}, \quad |z| > r, \quad \mathrm{Re}\, z < r; \\ E_2(z) &= E_0 (1 + \Delta_{12}) \left\{ 1 + r^2 \sum_{k=1}^{\infty} \left(\frac{(\Delta_{12}\Delta_{13})^k}{k^2} \left(z - \frac{k+1}{k} r \right)^{-2} \right] \right\}, \quad |z| < r; \\ E_3(z) &= \frac{E_0 + \Delta_{13}\bar{E}_0}{1 - \Delta_{13}} - \bar{E}_0 (1 + \Delta_{13}) \frac{r^2}{\Delta_{13}} \sum_{k=1}^{\infty} \left[\frac{(\Delta_{12}\Delta_{13})^k}{k^2} \left(z - \frac{k-1}{k} r \right)^{-2} \right], \quad \mathrm{Re}\, z > r. \end{aligned} \tag{3.1}$$

As can be seen, the solution is represented as a series whose components are expressions for plane dipoles. One series of dipoles is located on the segment [0, r) of the real axis, and the other at its mirror point relative to the point x = r on the segment (r, 2r]. As the order number k increases, the dipole coordinates concentrate at the point x = r (the tangent point of the inclusion and the boundary separating the different dielectrics).

Particular Solutions. Several simple solutions for two-component systems can be obtained from the general Eqs. (3.1).

1. If $\varepsilon_1 = \varepsilon_3$ ($\Delta_{13} = 0$), Eqs. (3.1) are transformed to

$$E_1(z) = E_3(z) = E_0 - \frac{\bar{E}_0 \Delta_{12} r^2}{z^2}, \quad E_2(z) = E_0(1 + \Delta_{12}).$$

This is the standard solution to the problem of the electric field of a dielectric cylinder of permeability ε_2 immersed in an unbounded dielectric medium of permeability ε_1, in an external homogeneous electric field E_0.

2. If $\varepsilon_1 = \varepsilon_2$ ($\Delta_{12} = 0$), then the solution from Eqs. (3.1) is

$$E_1(z) = E_0, \quad \mathrm{Re}\, z < r; \quad E_3(z) = \frac{E_0 + \Delta_{13}\bar{E}_0}{1 - \Delta_{13}}, \quad \mathrm{Re}\, z > r.$$

These expressions define the homogeneous electric field in a medium consisting of two dielectrics, which fill half spaces separated by a plane. The resultant expressions coincide with Eqs. (2.11), as they should.

Approximate Solutions. The dipole moments in Eqs. (3.1) contain parameters $(\Delta_{12}\Delta_{13})^k$ and $1/k^2$, which decrease rapidly in absolute value as k increases. Therefore the terms of the dipole series in Eqs. (3.1) die off rapidly with increasing order. In practical calculations this allows a small number of terms to be retained in the series. In this case the attainable accuracy can be estimated as follows.

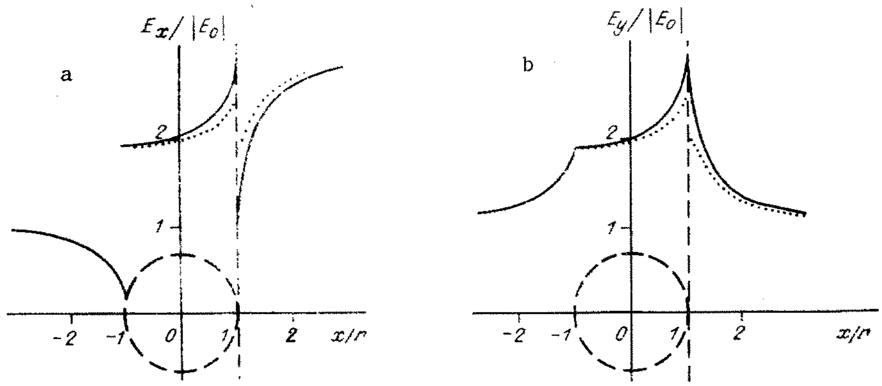

Fig. 2

Let $E_\ell(z) = E_{\ell m}(z) + Q_{\ell m}(z)$, $\ell = \overline{1, 3}$, where the $E_{\ell m}(z)$ are partial sums of m terms of the corresponding series (3.1). Then the following estimates are valid for the other terms $Q_{\ell m}(z)$ in the region $|z - r| < r$:

$$
\begin{aligned}
|Q_{1l}(z)| &\leqslant |E_0| \frac{|\Delta_{12}|^{l+1} |\Delta_{13}|^l (1+|\Delta_{13}|)}{1-|\Delta_{12}\Delta_{13}|}, \\
|Q_{2l}(z)| &\leqslant |E_0| \frac{|\Delta_{12}\Delta_{13}|^{l+1}(1+\Delta_{12})}{1-|\Delta_{12}\Delta_{13}|}, \\
|Q_{3l}(z)| &\leqslant |E_0| \frac{|\Delta_{12}|^{l+1} |\Delta_{13}|^l (1+\Delta_{13})}{1-|\Delta_{12}\Delta_{13}|}, \quad |z-r|<r.
\end{aligned} \tag{3.2}
$$

A coarser estimate away from the point z = r has the form

$$
\begin{aligned}
|Q_{1l}(z)| &< |E_0| \left(1+\frac{4}{3|\Delta_{13}|}\right) |\Delta_{12}\Delta_{13}|^{l+1} \delta, \\
|Q_{2l}(z)| &< |E_0| (1+\Delta_{12}) |\Delta_{12}\Delta_{13}|^{l+1} \delta, \\
|Q_{3l}(z)| &< |E_0| (1+\Delta_{13}) |\Delta_{12}|^{l+1} |\Delta_{13}|^l \delta, \qquad |z-r| \geqslant r.
\end{aligned} \tag{3.3}
$$

where

$$
\delta = \min\left(\frac{1}{l}, \frac{1}{(l+1)^2 (1-|\Delta_{12}\Delta_{13}|)}\right). \tag{3.4}
$$

For qualitative analysis of electric processes in the system (and sometimes also for obtaining required quantitative relationships), it is sufficient to use the first approximation of Eqs. (3.1), which consider dipoles only at the points z = 0 and r = 2r:

$$
\begin{aligned}
E_{11}(z) &= E_0 - \frac{\overline{E}_0 \Delta_{12} r^2}{z^2} + \frac{E_0 \Delta_{12}\Delta_{13} r^2}{(z-2r)^2}, \quad |z|>r, \quad \operatorname{Re} z<r; \\
E_{21}(z) &= E_0 (1+\Delta_{12}) \left[1 + \frac{\Delta_{12}\Delta_{13} r^2}{(z-2r)^2}\right], \quad |z|<r; \\
E_{31}(z) &= \frac{E_0 + \Delta_{13}\overline{E}_0}{1-\Delta_{13}} - \overline{E}_0 \Delta_{12} (1+\Delta_{13}) \frac{r^2}{z^2}, \quad \operatorname{Re} z > r.
\end{aligned} \tag{3.5}
$$

The accuracy of this approximation can be determined from Eqs. (3.2)-(3.5) for $\ell = 1$ in the corresponding region.

The features in the electric field image in the system under study can be estimated from how this field changes along the x-axis. The corresponding behaviors for the relative electric field strength $E_{x,y}/|E_0|$, where $|E_0|$ is the absolute value of the external electric field strength function, are shown in Fig. 2 for the case where the relative dielectric strengths of an inhomogeneous material are $\varepsilon_r = \{\varepsilon_{r1}, \varepsilon_{r2}, \varepsilon_{r3}\} = \{9, 1, 3\}$ (the curves are constructed for relative dielectric strengths $\varepsilon_r = \varepsilon/\varepsilon_0$, where ε_0 is the dielectric constant). In this example the external electric field strength function E_0 has a component only along the x-axis ($\theta = 0$, where θ is the angle between E_0 and the x-axis) in Fig. 2a, and along the y-axis in Fig. 2b ($\theta = \pi/2$). Under these conditions, the relationships

$$E_x = E_x(x), \quad E_y = 0 \quad \text{for } \theta = 0;$$
$$E_x = 0, \quad E_y = E_y(x) \quad \text{for } \theta = \pi/2.$$

are valid due to the structural symmetry of the system on the x-axis.

The solid curves in Fig. 2 are constructed from the exact Eqs. (3.1), while the dotted ones show the same functions calculated using the approximate Eqs. (3.5); the dashed curves show the system boundaries.

When we examine the curves in Fig. 2, our attention is turned to the sharp change in the electric field strength near the point where the cylindrical body is tangent to the boundary plane between the dielectrics. Physically, this behavior of the electric field strength functions is explained by the inhomogeneous structure of the material at the contact zone between the dielectric materials with different permeabilities. From a theoretical viewpoint, the features in the electric field in this zone can be explained by the fact that the tangent points of the cylindrical body with the plane boundary is a concentration point of dipoles in the series used to represent the solution to the boundary problem.

In order to obtain more exact calculations directly next to the point $z = r$, consideration of (3.2) shows that we must take a larger number of terms in the series (3.1). Outside this region the exact and approximate values almost coincide, according to Eqs. (3.3) and (3.4).

One more characteristic feature of the electric field in the system should be noted. We know that the field is also homogeneous inside an isolated dielectric isotropic cylinder located in a transverse homogeneous electric field. This trend arises in this case – in most of the cylindrical body the electric field is close to homogeneous and becomes significantly inhomogeneous only near its tangent point with the boundary plane.

4. Forces in the Electric Field. According to the initial conditions, there are no free charges in this inhomogeneous system. It is also assumed that no electrostriction effects occur in the dielectric materials. In this case polarization processes are the only reason that forces arise in the system. The force density vector $\mathbf{f}$ in inhomogeneous dielectrics is determined by the formula [1, 2]

$$\mathbf{f} = \frac{1}{2} E^2 \operatorname{grad} \varepsilon. \tag{4.1}$$

If the dielectric permeability does not change smoothly but is piecewise homogeneous, as it is in this system, then the force f acts only on surfaces separating dielectric materials with different permeabilities. The force vector $\mathbf{f}$ coincides with the normal to the boundary surface and is expressed by [2]:

$$\mathbf{f} = \frac{1}{2}\mathbf{n}(\varepsilon_{(-)} - \varepsilon_{(+)})\left[E^2_{(-)t} + \frac{\varepsilon_{(-)}}{\varepsilon_{(+)}}E^2_{(-)n}\right] = \frac{1}{2}\mathbf{n}(\varepsilon_{(-)} - \varepsilon_{(+)})\left[E^2_{(+)t} + \frac{\varepsilon_{(+)}}{\varepsilon_{(-)}}E^2_{(+)n}\right]. \tag{4.2}$$

Here $\mathbf{n}$ is the unit normal vector to the boundary surface and is pointed into the material whose parameters are noted by the subscript (+); $\varepsilon_{(+)}$ and $\varepsilon_{(-)}$ are the dielectric permeabilities of the adjacent dielectrics, and $E_{(\pm)n}$ and $E_{(\pm)t}$ are the normal and tangential components of the electric field strength on the boundary. According to Eq. (4.2), the electric field strength vector must be known on the boundaries in order to determine the vector $\mathbf{f}$. This value of the field is known from Eq. (3.1); here, in order to go from the terms E(z) used to represent the solution to the vector E, which figures in Eq. (4.2), it is necessary to make use of the simple relationships

$$E_n = \operatorname{Re}(E\bar{n}) = \operatorname{Re}(En) = (En + \bar{E}\bar{n})/2,$$
$$E_t = -\operatorname{Im}(E\bar{n}) = \operatorname{Im}(En) = (En - \bar{E}\bar{n})/2i,$$

where n is the unit normal, and the bars indicate the complex conjugate as before.

In order to illustrate how the forces act on the system, Figs. 3 and 4 show the results of calculating the force density distribution on the boundaries, when the electric permeabilities are $\varepsilon_r = \{9, 1, 3\}$ and $\varepsilon_r = \{1, 5, 9\}$, and the function E_0 has the argument $\theta = \pi/4$. The figures show curves of the force density as relative quantities

$$f_* = f/f_0, \quad f_0 = \varepsilon_0 |F_0|^2. \tag{4.3}$$

Depending on the relationship between the dielectric permeabilities ε_2 of the inclusion and ε_1 of the surrounding material, the inclusion is acted on by compressive forces (for

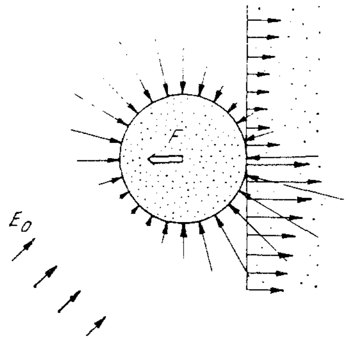

Fig. 3

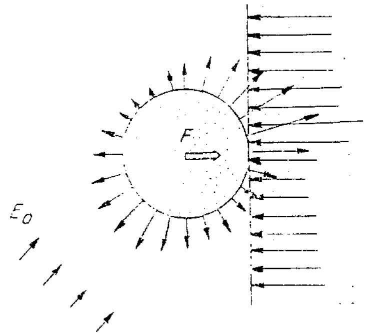

Fig. 4

$\varepsilon_1 > \varepsilon_2$, Fig. 3) or tensile forces (for $\varepsilon_1 < \varepsilon_2$, Fig. 4). As can be seen, these forces tend to pull the inclusion along the vector of the external electric field strength.

The plane boundary between the dielectrics causes a nonuniformity in the distribution of the force density f on the surface of the inclusion, which leads to a integral force

$$F = \int_0^{2\pi} f_* \, d\theta, \tag{4.4}$$

where the force f_* is defined by Eqs. (4.2) and (4.3).

The force F is directed normal to the plane boundary between the different dielectrics (with positive or negative sign) in all cases (for any orientation of the external electric field and for any dielectric material parameters). If $\varepsilon_1 > \varepsilon_2$ (as in Fig. 3), then the integral force acts to tear the inclusion from the boundary plane. However, if $\varepsilon_1 < \varepsilon_2$ (see Fig. 4), then the force F is pointed in the opposite direction and its action tends to trap the inclusion at the boundary surface.

Analysis of Eqs. (3.1) and (4.1)-(4.4) shows that the integral force becomes larger, the larger the difference between the dielectric permeabilities of the materials that comprise the inhomogeneous system. This is confirmed by actual calculations.

The inhomogeneous system models electrical processes under several electrophysical conditions. Actual conditions close to the modeled ones exist, for example, in systems where the solid dielectrics are bound by liquids or gases which contain included impurities.

Thanks to the analytic solution of the field problem, the force problem is simplified and requires only simple algebra. Calculations, done over a wide range of system parameters, made it possible to bring out all the features of the forces, which reduce to the following:

1. The force density vector f at the boundary of the different dielectrics as always pointed to the dielectric which has the smaller permeability.
2. The forces on the surface of the inclusion are distributed such that they act (by compression or tension) to pull the inclusion along the external electric field vector E_0.
3. The integral force vector F always acts normal to the plane boundary between the dielectric materials (which have a positive or negative sign) and is independent of the direction of the external electric field E_0.
4. The integral force vector F is directed opposite to the force density vector f on the plane boundary between the dielectric materials.
5. The integral force F increases as the values of the relative parameters Δ_{12} and Δ_{13} increase.

It should be noted that conclusions 1 and 2 follow from the overall physical assumptions; in particular they follow from Eqs. (4.2). Details of how the forces act are not that obvious and their determination requires a complete calculation of the electric field in the system and then a subsequent calculation of the force density distribution at the boundaries.

Appendix. Below we give an exact solution to the boundary problem (2.10) and (2.11), which are used to calculate the electric field in this system. The method of conformal mapping is used for the solution, which reduces the problem to a single functional equation.

This can be solved in closed form, which makes it possible to obtain explicit values for the field in all components of the system.

The linear fractional function

$$z = T(\zeta) = r\frac{\zeta+1}{\zeta-1} \quad (\zeta = \xi + i\eta) \tag{A.1}$$

is used to map the z plane into the ζ plane. Here the contours L_1 and L_2 transform to the straight lines $\lambda_1 = \{\zeta\colon \operatorname{Re}\zeta = 0\}$ and $\lambda_2 = \{\zeta\colon \operatorname{Re}\zeta = 1\}$, respectively, and the region S_p is mapped to the region Ω_p (p = 1, 2, 3), as shown in Fig. 1.

The reverse mapping of (A.1) is obtained by the function

$$\zeta = T^{-1}(z) = \frac{z+r}{z-r}. \tag{A.2}$$

The boundary conditions (2.10) for the function $f(\zeta) - E(T(\zeta))$ take the form

$$\begin{aligned} &(1+\Delta_{12}) f_1(\tau) = f_2(\tau) - \Delta_{12}\left(\frac{r}{T(\tau)}\right)^2 \overline{f_2(\tau)}, \quad \tau \in \lambda_1; \\ &(1+\Delta_{13}) f_1(\tau) = f_3(\tau) - \Delta_{13}\overline{f_3(\tau)}, \quad \tau \in \lambda_2. \end{aligned} \tag{A.3}$$

One of the equations

$$f_1(1) = E_0, \quad f_3(1) = \frac{E_0 + \Delta_{13}\overline{E}_0}{1-\Delta_{13}}. \tag{A.4}$$

must be considered along with the boundary conditions. Equations (A.4) follow from (2.11); they result from each other and the second Eq. (A.3). The problem (A.3) and (A.4) must be solved from the class of functions which have only integrable singularities at a point infinitely far away $[T^{-1}(r) = \infty]$.

The function $f_1(\zeta)$ can be represented as

$$f_1(\zeta) = f_1^+(\zeta) + f_1^-(\zeta), \tag{A.5}$$

where the function $f_1^+(\zeta)$ [or $f_1^-(\zeta)$] is holomorphic on the half plane $\operatorname{Re}\zeta < 1$ [or $\operatorname{Re}\zeta > 0$]. In order to obtain a single-valued representation of (A.5) we take

$$f_1^+(1) = 0, \quad f_1^-(1) = E_0. \tag{A.6}$$

[with a consideration of (A.4)].

By using (A.5), Eq. (A.3) is transformed to the form

$$\begin{aligned} &(1+\Delta_{12}) f_1^+(\tau) - f_2(\tau) = -\Delta_{12}\left(\frac{r}{T(\tau)}\right)^2 \overline{f_2(\tau)} - (1+\Delta_{12}) f_1^-(\tau), \quad \tau \in \lambda_1; \\ &(1+\Delta_{13}) f_1^-(\tau) - f_3(\tau) = -\Delta_{13}\overline{f_3(\tau)} - (1+\Delta_{13}) f_1^+(\tau), \quad \tau \in \lambda_2. \end{aligned}$$

From this it follows from the principle of analytic continuation that the functions

$$\begin{aligned} &\Phi(\zeta) = \begin{cases} (1+\Delta_{12}) f_1^+(\zeta) - f_2(\zeta), & \operatorname{Re}\zeta \leqslant 0; \\ -(1+\Delta_{12}) f_1^-(\zeta) - \Delta_{12}\left(\frac{r}{T(\zeta)}\right)^2 \overline{f_2(-\overline{\zeta})}, & \operatorname{Re}\zeta \geqslant 0, \end{cases} \\ &\Psi(\zeta) = \begin{cases} (1+\Delta_{13}) f_1^-(\zeta) - f_3(\zeta), & \operatorname{Re}\zeta \geqslant 1; \\ -(1+\Delta_{13}) f_1^+(\zeta) - \Delta_{13}\overline{f_3(2-\overline{\zeta})}, & \operatorname{Re}\zeta \leqslant 1 \end{cases} \end{aligned} \tag{A.7}$$

are holomorphic in the ζ plane, because the function $r/T(\zeta) = (\zeta - 1)/(\zeta + 1)$ is holomorphic for $\operatorname{Re}\zeta > 0$. The functions (A.7) can have only integrable singularities at infinity; therefore, based on Liouville's theorem,

$$\Phi(\zeta) \equiv C_1 = \text{const}, \quad \Psi(\zeta) \equiv C_2 = \text{const}.$$

The constants C_1 and C_2 are determined by substituting Eqs. (A.4) and (A.6) into Eq. (A.7), which leads to the following result:

$$\Phi(\zeta) = -(1+\Delta_{12}) E_0, \quad \Psi(\zeta) = -\Delta_{13}\frac{\overline{E}_0 + \Delta_{13}E_0}{1-\Delta_{13}}. \tag{A.8}$$

Two pairs of identities can be obtained from Eqs. (A.1) and (A.8):

$$\begin{aligned} &f_2(\zeta)-(1+\Delta_{12})f_1^+(\zeta)\equiv(1+\Delta_{12})E_0, \qquad \operatorname{Re}\zeta\leqslant 0;\\ &(1+\Delta_{12})f_1^-(\zeta)+\Delta_{12}\left(\frac{r}{T(\zeta)}\right)^2\overline{f_2(-\bar{\zeta})}\equiv(1+\Delta_{12})E_0, \quad \operatorname{Re}\zeta\geqslant 0;\\ &f_3(\zeta)-(1+\Delta_{13})f_1^-(\zeta)\equiv\Delta_{13}\frac{\bar{E}_0+\Delta_{13}E_0}{1-\Delta_{13}}, \quad \operatorname{Re}\zeta\geqslant 1;\\ &\Delta_{13}\overline{f_3(2-\bar{\zeta})}+(1+\Delta_{13})f_1^+(\zeta)\equiv\Delta_{13}\frac{\bar{E}_0+\Delta_{13}E_0}{1-\Delta_{13}}, \qquad \operatorname{Re}\zeta\leqslant 1. \end{aligned} \tag{A.9}$$

According to Eqs. (A.5) and (A.9),

$$\begin{aligned} &f_1(\zeta)=f_1^+(\zeta)+f_1^-(\zeta), \qquad 0<\operatorname{Re}\zeta<1;\\ &f_2(\zeta)=(1+\Delta_{12})f_1^+(\zeta)+(1+\Delta_{12})E_0 \qquad \operatorname{Re}\zeta<0;\\ &f_3(\zeta)=(1+\Delta_{13})f_1^-(\zeta)+\Delta_{13}\frac{\bar{E}_0+\Delta_{13}E_0}{1-\Delta_{13}}, \qquad \operatorname{Re}\zeta>1. \end{aligned} \tag{A.10}$$

Consequently only two functions, $f_1^+(\zeta)$ and $f_1^-(\zeta)$, have to be found in order to find the functions $f_p(\zeta)$ ($p = 1, 2, 3$).

First we exclude the functions $f_2(\zeta)$ and $f_3(\zeta)$ from Eqs. (A.9). Use of the symmetry transformation relative to the lines λ_1 and λ_2, respectively, in the first and second pair of identities (A.9) transforms them to

$$\begin{aligned} &\overline{f_2(-\bar{\zeta})}-(1+\Delta_{12})\overline{f_1^+(-\bar{\zeta})}=(1+\Delta_{12})\bar{E}_0, \qquad \operatorname{Re}\zeta\geqslant 0;\\ &(1+\Delta_{12})f_1^-(\zeta)+\Delta_{12}\left(\frac{\zeta-1}{\zeta+1}\right)^2\overline{f_2(-\bar{\zeta})}=(1+\Delta_{12})E_0, \quad \operatorname{Re}\zeta\geqslant 0;\\ &f_3(\zeta)-(1+\Delta_{13})f_1^-(\zeta)=\Delta_{13}\frac{\bar{E}_0+\Delta_{13}E_0}{1-\Delta_{13}}, \qquad \operatorname{Re}\zeta\geqslant 1,\\ &\Delta_{13}f_3(\zeta)+(1+\Delta_{13})\overline{f_1^+(2-\bar{\zeta})}=\Delta_{13}\frac{E_0+\Delta_{13}\bar{E}_0}{1-\Delta_{13}}, \qquad \operatorname{Re}\zeta\geqslant 1. \end{aligned} \tag{A.11}$$

It follows from Eq. (A.11) that

$$\begin{aligned} &f_1^-(\zeta)+\Delta_{12}\left(\frac{\zeta-1}{\zeta+1}\right)^2\overline{f_1^+(-\bar{\zeta})}=E_0-\Delta_{12}\left(\frac{\zeta-1}{\zeta+1}\right)^2\bar{E}_0, \qquad \operatorname{Re}\zeta\geqslant 0;\\ &\Delta_{13}f_1^-(\zeta)+\overline{f_1^+(2-\bar{\zeta})}=\Delta_{13}E_0, \qquad \operatorname{Re}\zeta\geqslant 1. \end{aligned} \tag{A.12}$$

By excluding the function $f_1^-(\zeta)$ from (A.12) we obtain

$$\overline{f_1^+(2-\bar{\zeta})}-\Delta_{12}\Delta_{13}\left(\frac{\zeta-1}{\zeta+1}\right)^2\overline{f_1^+(-\bar{\zeta})}=\Delta_{12}\Delta_{13}\left(\frac{\zeta-1}{\zeta+1}\right)^2\bar{E}_0, \qquad \operatorname{Re}\zeta\geqslant 1.$$

Now, by replacing ζ by $2-\bar{\zeta}$ and going to complex conjugate quantities, we finally obtain

$$f_1^+(\zeta)=\Delta_{12}\Delta_{13}\left(\frac{\zeta-1}{\zeta-3}\right)^2[E_0+f_1^+(\zeta-2)], \qquad \operatorname{Re}\zeta\leqslant 1. \tag{A.13}$$

The expression (A.13) is a functional equation relative to the unknown function $f_1^+(\zeta)$.

Sequential substitution of

$$\begin{aligned} &f_1^+(\zeta-2)=\Delta_{12}\Delta_{13}\left(\frac{\zeta-3}{\zeta-5}\right)^2[E_0+f_1^+(\zeta-4)],\\ &f_1^+(\zeta-4)=\Delta_{12}\Delta_{13}\left(\frac{\zeta-5}{\zeta-7}\right)^2[E_0+f_1^+(\zeta-6)], \text{ etc.} \end{aligned}$$

into the right side of (A.13) leads at the n-th step to

$$f_1^+(\zeta)=E_0(\zeta-1)^2\sum_{k=1}^{n}\left\{\frac{(\Delta_{12}\Delta_{13})^k}{(\zeta-2k-1)^2}\right\}+(\Delta_{12}\Delta_{13})^n\left(\frac{\zeta-1}{\zeta-2n-1}\right)^2 f_1^+(\zeta-2n), \quad \operatorname{Re} z\leqslant 1. \tag{A.14}$$

As n increases without bound, the last term on the right side of Eq. (A.14) tends to zero. Actually, it follows from the second Eq. (A.10) that the function $f_1^+(\zeta)$ has an integrable singularity at infinity, which means that the estimate

$$|f_1^+(\zeta-2n)|<|\zeta-2n|^{\alpha},$$

is valid for sufficiently large n, where $\alpha < 1$; therefore, for $\mathrm{Re}\,\zeta \leqslant 1$,

$$\left|(\Delta_{12}\Delta_{13})^n\left(\frac{\zeta-1}{\zeta-2n-1}\right)^2 f_1^+(\zeta-2n)\right|<|\Delta_{12}\Delta_{13}|^n\frac{|\zeta-1|^2|\zeta-2n|^{\alpha}}{|\zeta-1-2n|^2}\to 0$$

as $n \to \infty$ for any finite value of ζ. Consequently, Eq. (A.14) tends to

$$f_1^+(\zeta)=E_0(\zeta-1)^2\sum_{k=1}^{\infty}\frac{(\Delta_{12}\Delta_{13})^k}{(\zeta-2k-1)^2},\qquad \mathrm{Re}\,z\leqslant 1 \tag{A.15}$$

in the limit as $n \to \infty$. Therefore, for these values of ζ, the series (A.15) is bounded by the converging numerical series

$$\sum_{k=1}^{\infty}\frac{(\Delta_{12}\Delta_{13})^k}{(2k)^2},$$

which thus provides one of the two desired functions.

The second unknown function $f_1^-(\zeta)$ is determined from the second Eq. (A.12). Substituting Eq. (A.15) into this equation gives

$$f_1^-(\zeta)=E_0-\frac{E_0}{\Delta_{13}}(\zeta-1)^2\sum_{k=1}^{\infty}\frac{(\Delta_{12}\Delta_{13})^k}{(\zeta+2k-1)^2},\qquad \mathrm{Re}\,z\geqslant 1. \tag{A.16}$$

Equations (A.15), (A.16), and (A.10) make it possible to write explicit expressions for the functions $f_p(\zeta)$, p = 1, 2, 3. Then, reverse mapping of (A.2) is used in Sec. 3 to find Eqs. (3.1) for the electric field.

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GENERATION OF ELECTRICAL PULSES DURING FORMATION AND DEVELOPMENT OF CURRENT-DRIVEN INSTABILITIES IN PLASMAS

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In a current-carrying circuit left to itself, the electromagnetic forces act to increase the inductance. This is a consequence of the general principle that a system evolves in the direction of reduced potential energy. Initially stored in the electromagnetic field, the potential energy is converted into internal and kinetic energy of moving conductors. A current-carrying circuit is unstable with respect to increasing inductance.

Convincing examples of practical devices that operate on this principle include z- and θ-pinches, rail guns, electrodynamic accelerators of plasmas and solids, plasma dynamic opening switches, etc. As will be shown below, instabilities with respect to increasing inductance are of considerable importance in plasma opening switches.

When a current-driven instability develops, an inductive emf that is controlled by the currents and voltages in the circuit appears on the portions of the circuit with increasing inductance. The possible generation of electrical pulses by changes in the inductance under the action of intrinsic currents has been discussed from this point of view [1]. This analysis was carried out for motion in specified plane and cylindrical geometries. It was shown that in a formal mathematical sense, the emf increases without bound in a z-pinch under these assumptions [1]. This method may be useful for generating voltages and interrupting currents.

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